

CONVEXITY FOR TWISTED CONJUGATION

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ABSTRACT. Let G be a compact, simply connected Lie group. If $\mathcal{C}_1, \mathcal{C}_2$ are two G -conjugacy classes, then the set of elements in G that can be written as products $g = g_1 g_2$ of elements $g_i \in \mathcal{C}_i$ is invariant under conjugation, and its image under the quotient map $G \rightarrow G/\text{Ad}(G) = \mathfrak{A}$ is a convex polytope. In this note, we will prove an analogous statement for *twisted conjugations* relative to group automorphisms. The result will be obtained as a special case of a convexity theorem for group-valued moment maps which are equivariant with respect to the twisted conjugation action.

1. INTRODUCTION

Let G be a compact connected Lie group, with maximal torus T , and let $\mathfrak{g}, \mathfrak{t}$ be their Lie algebras. Fix a positive Weyl chamber $\mathfrak{t}_+ \subseteq \mathfrak{t}$, and denote by $p: \mathfrak{g} \rightarrow \mathfrak{t}_+$ the quotient map, with fiber $p^{-1}(\xi) = \mathcal{O}_\xi$ the adjoint orbit of ξ . For any $r > 1$, the set

$$(1) \quad \{(\xi_1, \dots, \xi_r) \in \mathfrak{t}_+ \times \dots \times \mathfrak{t}_+ \mid \exists \zeta_i \in \mathcal{O}_{\xi_i}: \zeta_1 + \dots + \zeta_r = 0\}$$

is a convex polyhedron known as the *Horn cone*. Fixing ξ_1, \dots, ξ_{r-1} , the Horn cone describes the set of adjoint orbits contained in the sum of adjoint orbits $\mathcal{O}_{\xi_1} + \dots + \mathcal{O}_{\xi_{r-1}}$. For the case of $G = \text{U}(N)$, the projection $p(\zeta)$ signifies the set of eigenvalues of a Hermitian matrix ζ , hence the Horn cone thus describes the possible eigenvalues of sums of Hermitian matrices with prescribed eigenvalues. The defining inequalities for the $\mathfrak{u}(n)$ -Horn cone were obtained by Klyachko [14], who gave a description in terms of the Schubert calculus of the Grassmannian. This was extended to arbitrary compact groups by Berenstein-Sjamaar [7]. See Ressayre [20] and Vergne-Walter [23] for recent developments.

Suppose in addition that G is simply connected. Let $\mathfrak{A} \subseteq \mathfrak{t}_+$ be the Weyl alcove. Then \mathfrak{A} labels the set of conjugacy classes in G , in the sense that there is a quotient map $q: G \rightarrow \mathfrak{A}$, with fiber $q^{-1}(\xi) = \mathcal{C}_\xi$ the conjugacy class of $\exp(\xi)$. As observed in Meinrenken-Woodward [17, Corollary 4.13], the set

$$(2) \quad \{(\xi_1, \dots, \xi_r) \in \mathfrak{A} \times \dots \times \mathfrak{A} \mid \exists g_i \in \mathcal{C}_{\xi_i}: g_1 \dots g_r = e\}$$

is a convex polytope. Put differently, this polytope describes the conjugacy classes arising in products of a collection of prescribed conjugacy classes. In the case of $G = \text{SU}(n)$, it describes the possible eigenvalues of *products* of special unitary matrices with prescribed eigenvalues; these eigenvalue inequalities were determined, in terms of quantum Schubert calculus on flag manifolds, by Agnihotri-Woodward [1] and Belkale [5]. (See also Belkale-Kumar [6].) This was extended to general G by Teleman-Woodward [22].

In this note we will show that there are similar polytopes for *twisted* conjugations. Recall that the twisted conjugation action relative to a group automorphism $\kappa \in \text{Aut}(G)$ is the action

$$(3) \quad \text{Ad}_g^{(\kappa)}(a) = g a \kappa(g^{-1}).$$

As we will explain, it suffices to consider automorphisms κ defined by Dynkin diagram automorphisms. These automorphisms preserve \mathfrak{t} , with fixed point set \mathfrak{t}^κ , and there is a convex polytope (*alcove*) $\mathfrak{A}^{(\kappa)} \subseteq \mathfrak{t}^\kappa$ with a quotient map $q^{(\kappa)}: G \rightarrow \mathfrak{A}^{(\kappa)}$ whose fiber $(q^{(\kappa)})^{-1}(\xi) = \mathcal{C}_\xi^{(\kappa)}$ is the κ -twisted conjugacy class of $\exp(\xi)$.

Theorem 1.1. *Let $\kappa_1, \dots, \kappa_r$ be diagram automorphisms with $\kappa_r \circ \dots \circ \kappa_1 = 1$. Then the set*

$$(4) \quad \{(\xi_1, \dots, \xi_r) \in \mathfrak{A}^{(\kappa_1)} \times \dots \times \mathfrak{A}^{(\kappa_r)} \mid \exists g_i \in \mathcal{C}_{\xi_i}^{(\kappa_i)}: g_1 \cdot \dots \cdot g_r = e\}$$

is a convex polytope.

It would be interesting to obtain an explicit description of the defining inequalities of the polytopes (4). Note that these polytopes (4) arise if one considers products of conjugacy classes of *disconnected* compact Lie groups K ; indeed each conjugacy class of K is a finite union of twisted conjugacy classes of the identity component $G = K_0$.

We will obtain Theorem 1.1 as a special case of a convexity result for group-valued moment maps that are equivariant under *twisted conjugation*. Examples of such spaces are the twisted conjugacy classes, or components of moduli spaces of flat connections for disconnected groups on surfaces with boundary. We have (cf. Theorem 4.4):

Theorem 1.2. *Let (M, ω, Φ) be a compact, connected q -Hamiltonian G -space with a κ -twisted equivariant moment map $\Phi: M \rightarrow G$. Then the fibers of the moment map are connected, and the image*

$$\Delta(M) := q^{(\kappa)}(\Phi(M)) \subseteq \mathfrak{A}^{(\kappa)}$$

is a convex polytope.

In a very recent paper, Boalch and Yamakawa [10] independently considered twisted group-valued moment maps in the context of twisted wild character varieties, generalizing earlier results of Boalch [8, 9]. In particular, their work has a discussion of twisted moduli spaces, similar to Section 3.2. I also learned about a forthcoming article by Alex Takeda, using twisted group-valued moment maps in the setting of shifted symplectic geometry.

Acknowledgment. I am grateful to Tom Baird for discussions on twisted conjugations and twisted moduli spaces.

2. TWISTED CONJUGATION

We begin by reviewing some background material on twisted conjugation actions. References include Baird [4], Kac [13], Mohr dieck [18], Mohr dieck-Wendt [19], and Springer [21].

2.1. Twisted conjugation. Let $\text{Aut}(G)$ be the group of automorphisms of a Lie group G , and $\text{Inn}(G) \cong G/Z(G)$ the normal subgroup of inner automorphisms Ad_a , $a \in G$. The quotient group is denoted $\text{Out}(G) = \text{Aut}(G)/\text{Inn}(G)$. For $\kappa \in \text{Aut}(G)$, define the κ -twisted conjugation action as

$$\text{Ad}_g^{(\kappa)}(h) = g h \kappa(g^{-1}).$$

Its orbits $\mathcal{C} \subseteq G$ are called the κ -twisted conjugacy classes. In terms of the semi-direct product $G \rtimes \text{Aut}(G)$, the twisted conjugation action can be regarded as an ordinary conjugation,

$$(g, 1)(h, \kappa)(g^{-1}, 1) = (gh\kappa(g^{-1}), \kappa).$$

For this reason, we will sometimes use the notation $G\kappa$ for the space G , regarded as a G -space under κ -twisted conjugation. For later reference, we note that if κ_1, κ_2 are two automorphisms, then

$$(5) \quad \text{Ad}_g^{(\kappa_2\kappa_1)}(h_1h_2) = \text{Ad}_g^{(\kappa_1)}(h_1) \text{Ad}_{\kappa_1(g)}^{(\kappa_2)}(h_2)$$

for all $g, h_1, h_2 \in G$.

The differential of $\kappa \in \text{Aut}(G)$ at the group unit is an automorphism of the Lie algebra \mathfrak{g} , still denoted by κ . The generating vector fields for the κ -twisted conjugation action are $\xi_G = \kappa(\xi)^L - \xi^R$ for $\xi \in \mathfrak{g}$. In terms of *right* trivialization of the tangent bundle, we have $\xi_G(h) = (\text{Ad}_h \circ \kappa - I)\xi$. Hence, the Lie algebra of the stabilizer of $h \in G$ is

$$(6) \quad \mathfrak{g}_h = \ker(\text{Ad}_h \circ \kappa - I),$$

while the tangent space to the twisted conjugacy class $\mathcal{C} = \text{Ad}_G^{(\kappa)}(h)$ is

$$(7) \quad T_h\mathcal{C} = \text{ran}(\text{Ad}_h \circ \kappa - I),$$

in *right* trivialization $T_hG = \mathfrak{g}$.

Suppose $\kappa' = \text{Ad}_a \circ \kappa$ for some $a \in G$. Then the corresponding twisted conjugations are related by right multiplication $r_a: G \rightarrow G$:

$$r_a \circ \text{Ad}_g^{(\kappa')} = \text{Ad}_g^{(\kappa)} \circ r_a.$$

That is, $g \mapsto ga^{-1}$ defines a G -map $G\kappa \rightarrow G\kappa'$. In particular, if \mathcal{C} is a κ -twisted conjugacy class then $\mathcal{C}' = r_{a^{-1}}(\mathcal{C})$ is a κ' -twisted conjugacy class.

Example 2.1. Suppose $\kappa_1, \dots, \kappa_r \in \text{Aut}(G)$, and let \mathcal{C}_i be κ_i -twisted conjugacy classes. Then the subset

$$\mathcal{C}_1 \cdots \mathcal{C}_r := \{h_1 \cdots h_r \mid h_i \in \mathcal{C}_i\} \subseteq G$$

is invariant under $\kappa := \kappa_r \cdots \kappa_1$ -twisted conjugation. This follows by induction from (5). Let $\kappa'_i = \text{Ad}_{a_i} \circ \kappa_i$ for some $a_i \in G$, and put $\mathcal{C}'_i = r_{a_i^{-1}}(\mathcal{C}_i)$ and $\kappa' = \kappa'_r \cdots \kappa'_1$. Then the problem of finding $h_i \in \mathcal{C}_i$ with product $h_1 \cdots h_r$ in a prescribed κ -twisted conjugacy class \mathcal{C} is equivalent to a similar problem for the \mathcal{C}'_i .

To see this, let u_1, \dots, u_{r+1} be inductively defined as $u_{i+1} = a_i \kappa_i(u_i)$ with $u_1 = e$, and put $a = u_{r+1}$. Then $\kappa' = \text{Ad}_a \circ \kappa$, hence $\mathcal{C}' = r_{a^{-1}}(\mathcal{C})$ is a κ' -twisted conjugacy class. A straightforward calculation shows that if $h_i \in \mathcal{C}_i$ satisfy $h := h_1 \cdots h_r \in \mathcal{C}$, then the elements

$$h'_i = \text{Ad}_{u_i}^{(\kappa_i)}(h_i) a_i^{-1} \in \mathcal{C}'_i$$

have product $h' = ha^{-1} \in \mathcal{C}'$.

2.2. Diagram automorphisms. Let G be a compact and simply connected Lie group, with maximal torus T and Weyl group $W = N_G(T)/T$. Fix a positive Weyl chamber $\mathfrak{t}_+ \subseteq \mathfrak{t}$, with corresponding alcove $\mathfrak{A} \subseteq \mathfrak{t}_+$. The walls of the Weyl chamber are defined by the simple roots $\alpha_1, \dots, \alpha_l \in \mathfrak{t}^*$. Let $\alpha_i^\vee \in \mathfrak{t}$ be the simple coroots, and let $e_i, f_i \in \mathfrak{g}^\mathbb{C}$ be the Chevalley generators, for $i = 1, \dots, l$.

Consider an automorphisms of the Dynkin diagram, given by a bijection $i \mapsto i'$ of its set of vertices preserving all Cartan integers: $\langle \alpha_i, \alpha_j^\vee \rangle = \langle \alpha_{i'}, \alpha_{j'}^\vee \rangle$. Any diagram automorphism defines a unique Lie algebra automorphism $\kappa \in \text{Aut}(\mathfrak{g}^\mathbb{C})$ such that $\kappa(e_i) = e_{i'}$, $\kappa(f_i) = f_{i'}$. This automorphism preserves the real Lie algebra $\mathfrak{g} \subseteq \mathfrak{g}^\mathbb{C}$, and exponentiates to the Lie group G . We will refer to the resulting $\kappa \in \text{Aut}(G)$ itself as a diagram automorphism. Every element of $\text{Out}(G) = \text{Aut}(G)/\text{Inn}(G)$ is represented by a unique diagram automorphism, and the resulting splitting $\text{Out}(G) \hookrightarrow \text{Aut}(G)$ identifies

$$\text{Aut}(G) = \text{Inn}(G) \rtimes \text{Out}(G).$$

That is, any automorphism of G can be written as $\kappa' = \text{Ad}_a \circ \kappa$ with $a \in G$ and $\kappa \in \text{Out}(G)$. To understand the orbit structure of κ -twisted conjugation actions, it hence suffices to consider the case that $\kappa \in \text{Aut}(G)$ is a diagram automorphism. In particular, κ preserves T , with fixed point set $T^\kappa \subseteq G^\kappa$. Let \mathfrak{t}^κ , \mathfrak{t}_κ be the kernel and range of $\kappa|_{\mathfrak{t}} - I: \mathfrak{t} \rightarrow \mathfrak{t}$. Then \mathfrak{t}^κ is the Lie algebra of T^κ , and $\mathfrak{t}_\kappa = (\mathfrak{t}^\kappa)^\perp$ is the orthogonal space in \mathfrak{t} (relative to a W -invariant metric). Put $T_\kappa = \exp(\mathfrak{t}_\kappa)$. Then $T = T^\kappa T_\kappa$, with finite intersection

$$T^\kappa \cap T_\kappa.$$

Let $W^\kappa \subseteq W$ the subgroup of elements w whose action on \mathfrak{t} commutes with κ . For $a \in G$, denote by G_a the stabilizer under the κ -twisted adjoint action. For Propositions 2.2 and 2.3 below, see [19], [21], and references therein.

Proposition 2.2. *Let $\kappa \in \text{Aut}(G)$ be a diagram automorphism. Then:*

- (a) *The group G^κ contains T^κ as a maximal torus, with Weyl group W^κ . The intersection $\mathfrak{t}_+^\kappa = \mathfrak{t}^\kappa \cap \mathfrak{t}_+$ is a positive Weyl chamber for G^κ .*
- (b) *Every κ -twisted conjugacy class $\mathcal{C} \subseteq G$ intersects the torus T^κ in an orbit of the finite group $(T^\kappa \cap T_\kappa) \rtimes W^\kappa$. Here $T^\kappa \cap T_\kappa$ acts by multiplication on T^κ .*
- (c) *For all $a \in T^\kappa$, the stabilizer group G_a under the twisted conjugation action contains T^κ as a maximal torus.*

Let $\Lambda = \exp_T^{-1}(e) \subseteq \mathfrak{t}$ be the integral lattice of T . Since G is simply connected, it coincides with the coroot lattice of (G, T) . The fixed point set $\Lambda^\kappa \subseteq \mathfrak{t}^\kappa$ is the integral lattice of T^κ . It is contained in the lattice,

$$\Lambda^{(\kappa)} = \exp_{T^\kappa}^{-1}(T^\kappa \cap T_\kappa).$$

Proposition 2.3. *There is a unique closed convex polytope $\mathfrak{A}^{(\kappa)} \subseteq \mathfrak{t}_+^\kappa$, containing the origin, such that $G_{\exp \xi} = T^\kappa$ for elements $\xi \in \text{int}(\mathfrak{A}^{(\kappa)})$, and such that the map*

$$\mathfrak{A}^{(\kappa)} \xrightarrow{\exp} G \longrightarrow G/\text{Ad}_G^{(\kappa)}$$

is a bijection. Furthermore,

- (a) *The cone over $\mathfrak{A}^{(\kappa)}$ is \mathfrak{t}_+^κ .*

- (b) For each open face $\sigma \subseteq \mathfrak{A}^{(\kappa)}$, the stabilizer group $G_\sigma := G_{\exp \xi}$ of elements $\xi \in \sigma$ does not depend on ξ . The stabilizer groups satisfy $G_\sigma \supseteq G_\tau$ for $\sigma \subseteq \tau$.
- (c) The group $W_{\text{aff}}^{(\kappa)} = \Lambda^{(\kappa)} \rtimes W^\kappa$ is an affine reflection group, generated by reflections across the facets of $\mathfrak{A}^{(\kappa)}$, and having $\mathfrak{A}^{(\kappa)}$ as a fundamental domain.

Clearly, if $\kappa = 1$ then $\mathfrak{A}^{(\kappa)}$ is just the usual Weyl alcove, parametrizing the set of (untwisted) conjugacy classes in G . Note that in general, $\Lambda^{(\kappa)}$, $\mathfrak{A}^{(\kappa)}$ are different from the coroot lattice and alcove of $(\mathfrak{g}^\kappa, \mathfrak{t}^\kappa)$. The group $W_{\text{aff}}^{(\kappa)}$ is the Weyl group of the twisted affine Lie algebra defined by κ , see Kac [13].

2.3. Slices. The conjugation action of G on itself has distinguished slices, labeled by the faces of the alcove. We will generalize this fact to twisted conjugation actions.

Lemma 2.4. *Let $\kappa \in \text{Aut}(G)$, where G is compact. Let $\mathcal{C} \subseteq G$ be the κ -twisted conjugacy class of an element $a \in G^\kappa$. Then*

$$(8) \quad T_a G = T_a(G_a) \oplus T_a \mathcal{C}.$$

Proof. Pick an $\text{Aut}(G)$ -invariant inner product on \mathfrak{g} , defining a bi-invariant Riemannian metric on the Lie group G which is also invariant under κ . Since $\kappa(a) = a$, we have $a \in G_a$, and we obtain $T_a G_a = \mathfrak{g}_a$ in right trivialization. On the other hand, by (6) and (7) we have $T_a \mathcal{C} = \text{ran}(\text{Ad}_a \circ \kappa - I) = \mathfrak{g}_a^\perp$ in right trivialization. Since the two spaces are orthogonal, the Lemma follows. \square

Using again that $\kappa(a) = a$, the twisted conjugation action of a on G restricts to the usual conjugation action on G_a . In particular, G_a is a $\text{Ad}_a^{(\kappa)}$ -invariant submanifold of G . The Lemma shows that any sufficiently small invariant open neighborhood of a in G_a is a slice for the twisted conjugation action.

If G is also simply connected, and κ is a diagram automorphism, there is a specific ‘largest’ slice, as follows. For any face $\sigma \subseteq \mathfrak{A}^{(\kappa)}$, let $\mathfrak{A}_\sigma^{(\kappa)}$ be the relatively open subset of $\mathfrak{A}^{(\kappa)}$ given as the union of faces $\tau \subseteq \mathfrak{A}^{(\kappa)}$ such that $\sigma \subseteq \tau$. Put

$$(9) \quad U_\sigma = \text{Ad}_{G_\sigma}^{(\kappa)} \exp(\mathfrak{A}_\sigma^{(\kappa)}),$$

a subset of $G_\sigma \subseteq G$.

Proposition 2.5. *The subset $U_\sigma \subseteq G_\sigma$ is open, and invariant under the twisted conjugation action of G_σ . The map*

$$(10) \quad G \times_{G_\sigma} U_\sigma \rightarrow G, [(g, a)] \mapsto \text{Ad}_g^{(\kappa)} a$$

is an embedding as an open subset of G . That is, U_σ is a slice for the twisted conjugation action.

Proof. Pick $\zeta \in \sigma$, and put $c = \exp \zeta$ so that $G_c = G_\sigma$. For all $\xi \in \mathfrak{t}^\kappa$ and $g \in G_\sigma$,

$$(11) \quad \text{Ad}_g^{(\kappa)} \exp(\xi) = \text{Ad}_g^{(\kappa)} (\exp(\xi - \zeta) c) = \text{Ad}_g (\exp(\xi - \zeta)) c.$$

It follows that $U_\sigma = U'_\sigma c$ where

$$U'_\sigma = \text{Ad}_{G_\sigma} \exp(\mathfrak{A}_\sigma^{(\kappa)} - \zeta).$$

Equation (11) also shows that for $\xi \in \mathfrak{A}_\sigma^{(\kappa)} \subseteq \mathfrak{t}^\kappa$, the stabilizer of $\exp(\xi)$ under the twisted conjugation action of G (which lies in G_σ , by definition of $\mathfrak{A}_\sigma^{(\kappa)}$) equals the stabilizer of $\exp(\xi - \zeta)$ under the usual conjugation action of G_σ . Consequently, $\mathfrak{A}_\sigma^{(\kappa)} - \zeta$ is a relatively open subset of an alcove of (G_σ, T^κ) . It follows that U'_σ is open in G_σ , and hence U_σ is open in G_σ .

We next show that the map (10) is injective. Thus suppose $\text{Ad}_g^{(\kappa)} a = \text{Ad}_{g'}^{(\kappa)} a'$, where $a, a' \in U_\sigma$ and $g, g' \in G$. Since a, a' are in the same twisted conjugacy class, there is a unique $\xi \in \mathfrak{A}_\sigma$ and elements $h, h' \in G_\sigma$ such that

$$a = \text{Ad}_h^{(\kappa)} \exp(\xi), \quad a' = \text{Ad}_{h'}^{(\kappa)} \exp(\xi).$$

We thus obtain

$$\text{Ad}_{gh}^{(\kappa)} \exp(\xi) = \text{Ad}_{g'h'}^{(\kappa)} \exp(\xi),$$

which implies $ghk = g'h'$ for some $k \in G_{\exp \xi} \subseteq G_\sigma$. Setting $u = h'k^{-1}h^{-1} \in G_\sigma$, we obtain $g' = gu^{-1}$, while $a' = \text{Ad}_u^{(\kappa)} a$. That is, $[(g, a)] = [(g', a')]$.

To complete the proof, it suffices to show that (10) has surjective differential. By equivariance, it is enough to verify this at elements $[(e, a)]$ with $a \in \exp(\mathfrak{A}_\sigma^{(\kappa)}) \subseteq T^\kappa$. The range of the differential of (10) at such a point contains $T_a G_\sigma + T_a \mathcal{C}$. Since $G_a \subseteq G_\sigma$, hence $T_a G_a \subseteq T_a G_\sigma$, Lemma 2.4 shows that this is all of $T_a G$. \square

3. Q-HAMILTONIAN SPACES

Let G be a Lie group, with an invariant inner product \cdot on its Lie algebra \mathfrak{g} , and let $\eta \in \Omega^3(G)$ be the bi-invariant closed 3-form

$$\eta = \frac{1}{12} \theta^L \cdot [\theta^L, \theta^L] = \frac{1}{12} \theta^R \cdot [\theta^R, \theta^R]$$

where $\theta^L, \theta^R \in \Omega^1(G, \mathfrak{g})$ are the left, right invariant Maurer-Cartan forms. Suppose $\kappa \in \text{Aut}(G)$ is an automorphism. It will be convenient to denote the group G , viewed as a G -manifold under κ -twisted conjugation, by $G\kappa$.

3.1. $G\kappa$ -valued moment maps. A q -Hamiltonian G -space with $G\kappa$ -valued moment map is a G -manifold M , together with an G -invariant 2-form ω and a G -equivariant smooth map $\Phi: M \rightarrow G\kappa$. These are required to satisfy the following axioms:

- (a) $d\omega = -\Phi^* \eta$,
- (b) $\iota(\xi_M) \omega = -\frac{1}{2} \Phi^*(\kappa(\xi) \cdot \theta^L + \xi \cdot \theta^R)$,
- (c) $\ker(\omega) \cap \ker(T\Phi) = 0$.

These axioms generalize the G -valued moment maps from [3]. In terms of equivariant de Rham forms, the first two properties may be combined into a single condition $d_G \omega = -\Phi^* \eta_G^{(\kappa)}$, where

$$\eta_G^{(\kappa)}(\xi) = \eta - \frac{1}{2}(\kappa(\xi) \cdot \theta^L + \xi \cdot \theta^R).$$

is a closed equivariant 3-form on $G\kappa$.

Example 3.1 (Twisted conjugacy classes). κ -twisted conjugacy classes $\mathcal{C} \subseteq G$ are q -Hamiltonian G -spaces, with the $G\kappa$ -valued moment map given as the inclusion. The 2-form is uniquely determined by the moment map condition (b), and is given by

$$\omega(\xi_{\mathcal{C}}, \tau_{\mathcal{C}}) = \frac{1}{2}((\text{Ad}_\phi \circ \kappa) - (\text{Ad}_\phi \circ \kappa)^{-1}) \xi \cdot \tau.$$

Note that the twisted conjugacy classes can be odd-dimensional. For example, in the case of $G = \mathrm{SU}(3)$ with κ given by complex conjugation, the generic stabilizer under twisted conjugation is a circle, and hence the generic twisted conjugacy classes are 7-dimensional.

Example 3.2 (Twisted moduli spaces). These are associated to any compact oriented surface with boundary, with marked points on the boundary components, with a prescribed homomorphism from the fundamental groupoid into $\mathrm{Aut}(G)$. This will be discussed in Section 3.2 below.

Example 3.3. Further examples are created by *fusion*: Suppose M_i for $i = 1, 2$ are two q-Hamiltonian G -spaces with $G\kappa_i$ -valued moment map. Then $M_1 \times M_2$ with the new G -action $g.(m_1, m_2) = (g.m_1, \kappa_1(g).m_2)$ and the 2-form

$$\omega = \omega_1 + \omega_2 + \frac{1}{2}\Phi_1^*\theta^L \cdot \Phi_2^*\theta^R$$

becomes a q-Hamiltonian G -space with $G\kappa_2\kappa_1$ -valued moment map $\Phi_1\Phi_2$. Properties (a) and (b) may be verified directly; for property (c) it is best to use the Dirac-geometric approach as in Remark 3.5.

For example, if \mathcal{C} is a κ -twisted conjugacy class in G , and M is a q-Hamiltonian G -space with (non-twisted) G -valued moment map, then the fusion product $M \times \mathcal{C}$ is a q-Hamiltonian G -space with $G\kappa$ -valued moment map. Also, if $\mathcal{C}_i \subseteq G$ are κ_i -twisted conjugacy classes, for $i = 1, \dots, r$, then their fusion product $\mathcal{C}_1 \times \dots \times \mathcal{C}_r$ is a q-Hamiltonian space with $G\kappa$ -valued moment map, where $\kappa = \kappa_r \cdots \kappa_1$. See example 2.1.

Remark 3.4. Let $L^{(\kappa)}G$ be the twisted loop group, consisting of paths $g: \mathbb{R} \rightarrow G$ with the property that $g(t+1) = \kappa(g(t))$ for all t . There is a notion of Hamiltonian $L^{(\kappa)}G$ -space generalizing that of a Hamiltonian LG -space [17], and by the same proof as for $\kappa = 1$ [3] one sees that there is a 1-1 correspondence between Hamiltonian $L^{(\kappa)}G$ -spaces with proper moment maps and κ -twisted q-Hamiltonian G -spaces.

Remark 3.5. The definition of $G\kappa$ -valued moment maps has a Dirac-geometric interpretation, similar to [11] and [2]. Using the notation from [2], let $\mathbb{A} = \mathbb{T}G_\eta$ be the standard Courant algebroid over G , with the Courant bracket twisted by the closed 3-form η . It has a canonical trivialization $\mathbb{A} = G \times (\bar{\mathfrak{g}} \oplus \mathfrak{g})$, where $\bar{\mathfrak{g}}$ stands for \mathfrak{g} with the opposite metric. Any Lagrangian Lie subalgebra $\mathfrak{s} \subseteq \bar{\mathfrak{g}} \oplus \mathfrak{g}$ defines a Dirac structure $E_{\mathfrak{s}} = G \times \mathfrak{s} \subseteq \mathbb{A}$. Taking \mathfrak{s} to be the diagonal, one obtains the Cartan-Dirac structure $E_{\Delta} = E$. Taking $\mathfrak{s} = \{(\xi, \kappa(\xi)) \mid \xi \in \mathfrak{g}\}$ for $\kappa \in \mathrm{Aut}(G)$, one obtains a Dirac structure $E_{\mathfrak{s}} = E^{(\kappa)}$ generalizing the Cartan-Dirac structure. As a Lie algebroid, it is the action Lie algebroid for the κ -twisted conjugation action. For a q-Hamiltonian space (M, ω, Φ) with $G\kappa$ -valued moment map, the pair (Φ, ω) defines a full morphism of Manin pairs, $(\mathrm{T}M, TM) \dashrightarrow (\mathbb{A}, E^{(\kappa)})$. Conversely, such a morphism defines a \mathfrak{g} -action on M for which the underlying map $\Phi: M \rightarrow G$ is equivariant, and it also defines an invariant 2-form on M satisfying the axioms above. The Dirac-geometric approach explains many of the properties of $G\kappa$ -valued moment maps; for example the fusion construction finds a conceptual explanation in terms of a Dirac morphism

$$(\mathrm{Mult}_G, \varsigma): (\mathbb{A}, E^{(\kappa_1)}) \times (\mathbb{A}, E^{(\kappa_2)}) \rightarrow (\mathbb{A}, E^{(\kappa_2\kappa_1)})$$

with the 2-form $\varsigma = \frac{1}{2} \mathrm{pr}_1^* \theta^L \cdot \mathrm{pr}_2^* \theta^R$. See [2, Section 4.4].

3.2. Twisted moduli spaces. Let $\Sigma = \Sigma_h^r$ be a compact, connected, oriented surface of genus h with $r > 0$ boundary components, and let $\mathcal{V} = \{x_1, \dots, x_r\}$ be a collection of base points on the boundary components, $x_i \in (\partial\Sigma)_i \cong S^1$. Let

$$\pi_1(\Sigma, \mathcal{V}) \rightrightarrows \mathcal{V}$$

denote the fundamental groupoid, consisting of homotopy classes λ of paths for which both the initial point $s(\lambda)$ and the end point $t(\lambda)$ are in \mathcal{V} . Suppose we are given a groupoid homomorphism (‘twist’)

$$\kappa \in \text{Hom}(\pi_1(\Sigma, \mathcal{V}), \text{Aut}(G)).$$

Such a κ may be obtained by assigning elements of $\text{Aut}(G)$ to a system of free generators of the fundamental groupoid, and extending by the homomorphism property. Let

$$(12) \quad M = \text{Hom}_\kappa(\pi_1(\Sigma, \mathcal{V}), G)$$

be the space of κ -twisted homomorphisms, consisting of maps $\lambda \mapsto \phi_\lambda$ such that

$$\phi_{\lambda_1 \circ \lambda_2} = \phi_{\lambda_1} \kappa_{\lambda_1}(\phi_{\lambda_2})$$

whenever $s(\lambda_1) = t(\lambda_2)$. (The space M may be regarded as a certain moduli space of flat connections.)¹ The group $\text{Map}(\mathcal{V}, G) = G \times \dots \times G$ act on the space (12) as

$$(g \cdot \phi)_\lambda = g_{t(\lambda)} \phi_\lambda \kappa_\lambda(g_{s(\lambda)}^{-1}).$$

Let $\kappa_1, \dots, \kappa_r \in \text{Aut}(G)$ be the values of κ on the oriented boundary loops $\lambda_1, \dots, \lambda_r$. Then M is a q-Hamiltonian G^r -space, with a $G\kappa_1 \times \dots \times G\kappa_r$ -valued moment map Φ given by evaluation on boundary loops. We won’t describe the 2-form here, since for the case that κ takes values in diagram automorphisms it may be regarded as a component of the moduli space of flat $G \rtimes \text{Out}(G)$ -bundles – see Section 3.4 below.

Remark 3.6. This construction also gives new examples of *non-twisted* q-Hamiltonian spaces. For example, take $\Sigma = \Sigma_1^1$ be the surface of genus 1 with one boundary component. Its fundamental group(oid) has free generators α, β , with the boundary loop given as $\alpha\beta\alpha^{-1}\beta^{-1}$. Attach an automorphism $\sigma \in \text{Aut}(G)$ to β , and 1 to α , and extend to a homomorphism κ as above. Then the corresponding M is $G \times G$, with elements (a, b) corresponding to holonomies along α, β . The group G acts on a by conjugation and on b by κ -twisted conjugation. The boundary holonomy is a G -valued moment map

$$(a, b) \mapsto ab\kappa(a^{-1})\kappa(b^{-1}),$$

a twisted group commutator.

Remark 3.7. Let K be a disconnected group with identity component $G = K_0$. The space $\text{Hom}(\pi_1(\Sigma, \mathcal{V}), K)$ is a moduli space of flat K -bundles over Σ , with framings at the base points. This space is disconnected, in general. The conjugation action of K on its identity component defines a group homomorphism $K \rightarrow \text{Aut}(G)$. Hence, any element $x \in \text{Hom}(\pi_1(\Sigma, \mathcal{V}), K)$ determines an element $\kappa \in \text{Hom}(\pi_1(\Sigma, \mathcal{V}), \text{Aut}(G))$, and the connected component containing x is identified with a component of $\text{Hom}_\kappa(\pi_1(\Sigma, \mathcal{V}), G)$.

¹Alternatively, Hom_κ is the lift of κ to the space $\widetilde{M} = \text{Hom}(\pi_1(\Sigma, \mathcal{V}), G \rtimes \text{Aut}(G))$.

The example giving rise to the convex polytope in Theorem 1.1 is obtained from the r -holed sphere $\Sigma = \Sigma_0^r$. Here $\pi_1(\Sigma, \mathcal{V})$ is freely generated by $\lambda_1, \dots, \lambda_{r-1}$, represented by oriented boundary loops based at x_1, \dots, x_{r-1} , together with μ_1, \dots, μ_{r-1} represented by non-intersecting paths connecting these points to x_r . The element λ_r represented by the remaining boundary loop satisfies

$$(13) \quad \prod_{i=1}^{r-1} \mu_i \lambda_i \mu_i^{-1} = \lambda_r^{-1}.$$

Given $\kappa \in \text{Hom}(\pi_1(\Sigma, \mathcal{V}), \text{Aut}(G))$, we denote by κ_i the images of the λ_i 's, and by σ_i the images of the μ_i 's. Then

$$(14) \quad \prod_{i=1}^{r-1} \sigma_i \kappa_i \sigma_i^{-1} = \kappa_r^{-1}.$$

We find $\text{Hom}_\kappa(\pi_1(\Sigma, \mathcal{V}), G) = G^{2r-2}$, consisting of tuples $(d_1, \dots, d_{r-1}, a_1, \dots, a_{r-1})$, where d_i are holonomies attached to the λ_i , and a_i are attached to the μ_i . The holonomy d_r around the r -th boundary loop is determined from

$$(15) \quad \prod_{i=1}^{r-1} (a_i, \sigma_i)(d_i, \kappa_i)(a_i, \sigma_i)^{-1} = (d_r, \kappa_r)^{-1}.$$

Lemma 3.8. *Let $\kappa_1, \dots, \kappa_r$ be holonomies attached to the boundaries of Σ_0^r , with $\kappa_r \kappa_{r-1} \cdots \kappa_1 = 1$. Then there is an extension to a homomorphism $\kappa \in \text{Hom}(\pi_1(\Sigma, \mathcal{V}), \text{Aut}(G))$, in such a way that the moment map image of $M = \text{Hom}_\kappa(\pi_1(\Sigma, \mathcal{V}), G)$ consists of all $(d_1, \dots, d_r) \in G^r$ for which there exists (g_1, \dots, g_r) with $g_i \in \text{Ad}_G^{(\kappa_i)}(d_i)$ and $\prod_{i=1}^r g_i = e$.*

Proof. Using the notation above, put $\sigma_1 = 1$, $\sigma_2 = \kappa_1^{-1}$, \dots , $\sigma_{r-1} = \kappa_1^{-1} \cdots \kappa_{r-2}^{-1}$. Equation (14) becomes the condition $\kappa_r \kappa_{r-1} \cdots \kappa_1 = 1$. Introducing

$$a'_1 = a_1, \quad a'_2 = \kappa_1(a_2), \quad a'_3 = \kappa_2(\kappa_1(a_3)), \quad \dots$$

the equation for the holonomies becomes

$$\prod_{i=1}^r a'_i d_i \kappa_i ((a'_i)^{-1}) = e$$

where we put $a'_r = e$. That is $\prod g_i = e$ where

$$g_i = a'_i d_i \kappa_i ((a'_i)^{-1}) \in \text{Ad}_G^{(\kappa_i)}(d_i).$$

The moment map for M is the map taking $(d_1, \dots, d_{r-1}, a_1, \dots, a_{r-1})$ to (d_1, \dots, d_r) , with d_r determined from the condition $\prod g_i = e$. \square

3.3. Basic properties of $G\kappa$ -valued moment maps. The following statement extends a well-known property of moment maps in symplectic geometry.

Proposition 3.9. *Let (M, ω, Φ) be a q -Hamiltonian G -space with $G\kappa$ -valued moment map. For all $m \in M$ we have*

$$\ker(T_m \Phi)^\omega = T_m(G \cdot m), \quad \text{ran}(\Phi^* \theta^R)_m = \mathfrak{g}_m^\perp.$$

(For any subspace $V \subseteq T_m M$, the notation V^ω means the set of all $v \in T_m M$ such that $\omega(v, w) = 0$ for all $w \in V$.)

Proof. In terms of $A = \text{Ad}_{\Phi(m)} \circ \kappa$, the moment map condition gives

$$(16) \quad \iota(\xi_M)\omega_m = -\frac{1}{2}((A + I)\xi) \cdot (\Phi^*\theta^R)_m.$$

In particular, for $\xi \in \mathfrak{g}_m$, we get that

$$\frac{1}{2}((A + I)\xi) \cdot (\Phi^*\theta^R)_m = 0.$$

But $\mathfrak{g}_m \subseteq \mathfrak{g}_{\Phi(m)} = \ker(A - I)$, so A acts as the identity on \mathfrak{g}_m . Hence we obtain $\xi \cdot (\Phi^*\theta^R)_m = 0$, proving $\text{ran}(\Phi^*\theta^R)_m \subseteq \mathfrak{g}_m^\perp$. On the other hand, it is immediate from the moment map condition that $\ker(T_m \Phi)^\omega \supseteq T_m(G \cdot m)$. Equality of both inclusions follows by a dimension count:

$$\begin{aligned} \dim(G \cdot m) &\leq \dim(\ker(T_m \Phi)^\omega) \\ &= \dim T_m M - \dim(\ker(T_m \Phi)) \\ &= \dim(\text{ran}(\Phi^*\theta^R)_m) \\ &\leq \dim \mathfrak{g}_m^\perp = \dim(G \cdot m). \end{aligned}$$

Here we used $\ker(\omega) \cap \ker(T\Phi) = 0$ for the first equality sign. \square

Proposition 3.10. *The map $\mathfrak{g} \rightarrow T_m M$ given by the infinitesimal action restricts to an isomorphism,*

$$\ker(\text{Ad}_{\Phi(m)} \circ \kappa + I) \xrightarrow{\cong} \ker(\omega_m).$$

Proof. Here, the Dirac-geometric viewpoint from Remark 3.5 is convenient. Let $\mathbb{T}G_\eta$ as in that remark. The subspace

$$E_1 = \{T\Phi(v) + \alpha \in \mathbb{T}G_\eta \mid v \in T_m M, \alpha \in T_{\Phi(m)}^* G, \Phi^*\alpha = \iota(v)\omega_m\}$$

is the ‘forward image’ of $T_m M \subseteq \mathbb{T}M = TM \oplus T^*M$ under the linear Dirac morphism $(T_m \Phi, \omega_m)$; in particular it satisfies $E_1 = E_1^\perp$. The axioms show that E_1 contains the space

$$E = \{\xi_G + \frac{1}{2}\theta^R \cdot (A + I)\xi \mid \xi \in \mathfrak{g}\}$$

(everything evaluated at $\Phi(m)$). Here ξ_G are the generating vector fields for the κ -twisted conjugation,

$$\xi_G = \kappa(\xi)^L - \xi^R = ((A - I)\xi)^R.$$

But it is easily checked that $E = E^\perp$, which together with $E \subseteq E_1$ implies $E_1 = E$. In particular, taking $\alpha = 0$ in the definition of E_1 we see that

$$(T_m \Phi)(\ker \omega_m) = \{\xi_G(\Phi(m)) \mid (A + I)\xi = 0\}.$$

Since $\ker(\omega_m) \cap \ker(T_m \Phi) = 0$, the map $T_m \Phi$ is injective on $\ker(\omega_m)$. Consequently, $\ker(\omega_m) = \{\xi_M(m) \mid (A + I)\xi = 0\}$. \square

3.4. Changing κ by inner automorphisms. Suppose $\kappa' = \text{Ad}_a \circ \kappa$, and let (M, ω, Φ) be a q -Hamiltonian G -space with $G\kappa$ -valued moment map. Then the manifold M with the same G -action and 2-form, but with a shifted moment map $\Phi' = r_{a^{-1}} \circ \Phi$, is a q -Hamiltonian G -space with $G\kappa'$ -valued moment map. For this reason, if G is compact and simply connected, it usually suffices to consider the case of diagram automorphism $\kappa \in \text{Out}(G)$. But for $\kappa \in \text{Out}(G)$, the q -Hamiltonian G -spaces with $G\kappa$ -valued moment map are simply q -Hamiltonian spaces with moment maps valued in the disconnected group $G \rtimes \text{Out}(G)$, whose image is contained in the component $G \times \{\kappa\}$. (The only wrinkle is that we only consider the action of the identity component G of this group, but this doesn't affect the theory from [3].) In this sense, the examples considered above are not new, at least for G compact and simply connected. For instance, in the fusion procedure 3.3, first apply the automorphism κ_1 to the second space, thus obtaining (M_2, ω_2, Φ'_2) with the new G -action $m \mapsto \kappa_1(g).m$, and a $G\kappa'$ -valued moment map $\Phi'_2 = \kappa_1^{-1} \circ \Phi_2$, where $\kappa' = \kappa_1^{-1} \kappa_2 \kappa_1$. Since

$$(\Phi_1, \kappa_1)(\kappa_1^{-1} \circ \Phi_2, \kappa_2) = (\Phi_1 \Phi_2, \kappa_2 \circ \kappa_1),$$

we recognize the fusion product 3.3 as a standard fusion product [3] for q -Hamiltonian G -spaces with $G \rtimes \text{Out}(G)$ -valued moment maps.

4. CONVEXITY PROPERTIES

We now turn to the convexity properties of $G\kappa$ -valued moment maps. The arguments are mostly straightforward adaptations of those in [17] and [15]. Throughout, we will assume that G is compact and simply connected, and that $\kappa \in \text{Aut}(G)$ is a diagram automorphism. We denote by $\mathfrak{A}^{(\kappa)} \subseteq \mathfrak{t}^\kappa$ the alcove, and by

$$q^{(\kappa)} : G \rightarrow \mathfrak{A}^{(\kappa)}$$

the quotient map, with fibers $(q^{(\kappa)})^{-1}(\xi)$ the κ -twisted conjugacy classes of $\exp(\xi)$. Recall from 2.3 the definition of the slices U_σ . Let κ_σ denote the restriction of κ to G_σ .

Proposition 4.1 (Cross-section theorem). *Let (M, ω, Φ) be a connected q -Hamiltonian G -space with $G\kappa$ -valued moment map. For any face $\sigma \subseteq \mathfrak{A}^{(\kappa)}$, the pre-image $Y_\sigma = \Phi^{-1}(U_\sigma)$ is a q -Hamiltonian $G_\sigma \kappa_\sigma$ -space, with the pullback of ω as the 2-form and the restriction of Φ as the moment map.*

The proof is parallel to the result for non-twisted q -Hamiltonian spaces, see [3], which in turn is a version of the cross-section theorem for Hamiltonian spaces, due to Guillemin-Sternberg [12] and Marle [16].

Recall that for any connected G -manifold M , the principal stratum M_{prin} is the set of all points whose stabilizer is subconjugate to any other stabilizer. It is connected, and open and dense in M .

Proposition 4.2. *Let (M, ω, Φ) be a connected q -Hamiltonian G -space with $G\kappa$ -valued moment map. Then:*

- (a) *The stabilizer G_m of any point $m \in M_{\text{prin}}$ is an ideal in $G_{\Phi(m)}$.*
- (b) *All points in $M_{\text{prin}} \cap \Phi^{-1}(\exp(\mathfrak{A}^{(\kappa)}))$ have the same stabilizer H .*

(c) *The image $q^{(\kappa)}(\Phi(M_{\text{prin}}))$ is a connected, relatively open subset of*

$$(x + \mathfrak{h}^\perp) \cap \mathfrak{A}^{(\kappa)},$$

where \mathfrak{h} is the Lie algebra of H , and x is any point of $q^{(\kappa)}(\Phi(M_{\text{prin}}))$.

Proof. The parallel statements for ordinary Hamiltonian G -spaces are proved in [15, Section 3.3]. In particular, if N is a connected Hamiltonian G -space, with moment map $\Psi: N \rightarrow \mathfrak{g}^*$, then for each $n \in N_{\text{prin}}$, the stabilizer G_n is an ideal in $G_{\Phi(n)}$, and the stabilizer $H = G_n$ of points in $N_{\text{prin}} \cap \Psi^{-1}(\mathfrak{t}_+^*)$ is independent of n . We will use cross-sections Y_σ to reduce to the Hamiltonian case. As noted in the proof of Proposition 2.5, the automorphism $\kappa_\sigma = \kappa|_{G_\sigma}$ is inner, and is given by $\text{Ad}_{a^{-1}}$ for any choice of $a \in \exp(\sigma)$. Hence, Y_σ becomes a q -Hamiltonian G_σ -space with (untwisted) G_σ -valued moment map $r_{a^{-1}} \circ \Phi_\sigma$. Furthermore, this then becomes an ordinary Hamiltonian G_σ -space with a moment map

$$\Phi_{0,\sigma}: Y_\sigma \rightarrow \mathfrak{g}_\sigma \cong \mathfrak{g}_\sigma^*, \quad m \mapsto \log(\Phi_\sigma(m)a^{-1}).$$

We conclude that for all $m \in (Y_\sigma)_{\text{prin}} = Y_\sigma \cap M_{\text{prin}}$, the stabilizer G_m is an ideal in the stabilizer of $\Phi_{0,\sigma}(m)$ under the adjoint action. The latter coincides with stabilizer of $\Phi(m) = \exp(\Phi_{0,\sigma}(m))a$ under twisted conjugation. Hence G_m is an ideal in $G_{\Phi(m)}$. Since the flow-outs of all the Y_σ 's under twisted conjugation cover M , this proves (a).

The map $M_{\text{prin}} \cap \Phi^{-1}(\exp(\mathfrak{A}^{(\kappa)})) \rightarrow M_{\text{prin}}/G$ is surjective, and has connected fibers $G_{\Phi(m)}.m = G_{\Phi(m)}/G_m$. Since the target of this map is connected, it follows that $M_{\text{prin}} \cap \Phi^{-1}(\exp(\mathfrak{A}^{(\kappa)}))$ is connected. Consider the decomposition of each Y_σ into its connected components Y_σ^i . Passing to the corresponding Hamiltonian G_σ -space as above, and using the general results for connected Hamiltonian spaces, we see that all points of $Y_\sigma^i \cap M_{\text{prin}} \cap \Phi^{-1}(\exp(\mathfrak{A}^{(\kappa)}))$ have the same stabilizer. Since the union of these sets, over all σ, i , covers $M_{\text{prin}} \cap \Phi^{-1}(\exp(\mathfrak{A}^{(\kappa)}))$, it follows that all points of this intersection have the same stabilizer, proving (b).

Each $q^{(\kappa)}(\Phi(M_{\text{prin}}) \cap Y_\sigma^i)$ is a connected, relatively open subset of $(x + \mathfrak{h}^\perp) \cap \mathfrak{A}_\sigma^{(\kappa)}$, for any choice of $x \in q^{(\kappa)}(\Phi(M_{\text{prin}}) \cap Y_\sigma^i)$. (Once again, this follows from the corresponding statement for Hamiltonian spaces, see [15, Section 3.3].) This implies (c). \square

Theorem 4.3 (Principal cross-section). *Let (M, ω, Φ) be a connected q -Hamiltonian G -space with $G\kappa$ -valued moment map. Then there exists a unique open face σ of $\mathfrak{A}^{(\kappa)}$ such that*

$$q^{(\kappa)}(\Phi(M)) \subseteq \overline{q^{(\kappa)}(\Phi(M)) \cap \sigma}.$$

(Equality holds if M is compact.) *Alternatively, σ is characterized as the smallest face such that the corresponding cross-section Y_σ satisfies $\Phi(Y_\sigma) \subseteq \exp(\sigma)$. This principal cross-section Y_σ is a connected q -Hamiltonian T^κ -space, with the restriction of Φ as the moment map, and*

$$M = \overline{G \cdot Y_\sigma}.$$

Proof. Using the notation from the previous proposition, let σ be the lowest dimensional face of $\mathfrak{A}^{(\kappa)}$ whose closure contains $(x + \mathfrak{h}) \cap \mathfrak{A}^{(\kappa)}$. Since $q^{(\kappa)}(\Phi(M_{\text{prin}}))$ is a relatively open subset of $(x + \mathfrak{h}) \cap \mathfrak{A}^{(\kappa)}$, its intersection with σ is non-empty. It follows that $q^{(\kappa)}(\Phi(M)) \cap \sigma = q^{(\kappa)}(\Phi(Y_\sigma))$. That is, $\Phi(Y_\sigma) \subseteq \exp(\sigma) \subseteq T^\kappa$, so that Y_σ may be regarded as a q -Hamiltonian T^κ -space, for the restriction of the moment map.

By construction, $G \cdot Y_\sigma = \Phi^{-1}((q^{(\kappa)})^{-1}(\sigma))$. The difference

$$(17) \quad M_{\text{prin}} - ((G \cdot Y_\sigma) \cap M_{\text{prin}}) = M_{\text{prin}} - (\Phi^{-1}((q^{(\kappa)})^{-1}(\sigma)) \cap M_{\text{prin}})$$

is the union over all $\Phi^{-1}((q^{(\kappa)})^{-1}(\tau)) \cap M_{\text{prin}}$ where τ ranges over proper faces of $\bar{\sigma}$. But those are submanifolds of codimension at least 3, hence removing them will not disconnect M_{prin} . Thus $(G \cdot Y_\sigma) \cap M_{\text{prin}}$ is connected, which implies that $G \cdot Y_\sigma = G \times_{G_\sigma} Y_\sigma$ is connected, and therefore Y_σ is connected. \square

Note that since the principal cross-section Y_σ is a q-Hamiltonian T^κ -space, it is in particular symplectic.

Theorem 4.4. *Let (M, ω, Φ) be a compact, connected q-Hamiltonian G -space with $G\kappa$ -valued moment map. Then the fibers of the moment map Φ are connected, and the image*

$$\Delta(M) := q^{(\kappa)}(\Phi(M)) \subseteq \mathfrak{A}^{(\kappa)}$$

is a convex polytope.

Proof. The principal cross-section $Y = Y_\sigma$ is a connected q-Hamiltonian T^κ -space, with the restriction $\Phi_Y = \Phi|_Y$ as its moment map. We can regard Y as an ordinary Hamiltonian T^κ -space, with a moment map $\Phi_{Y,0} = q^{(\kappa)} \circ \Phi_Y$ that is proper as a map to $\sigma \subseteq \mathfrak{t}^\kappa$.

Since σ is convex, [15, Theorem 4.3] shows that $\Phi_{Y,0}$ has connected fibers, and its image is a convex set of the form

$$q^{(\kappa)}(\Phi(Y)) = \Phi_{Y,0}(Y) = P \cap \sigma,$$

where P is some convex polytope in $\bar{\sigma}$. But then $q^{(\kappa)}(\Phi(M)) = \overline{q^{(\kappa)}(\Phi(Y))} = P$. Finally, if $x \in q^{(\kappa)}(\Phi(M))$, then the same argument as in [15] shows that for any open ball B around x , the pre-image $\Phi^{-1}((q^{(\kappa)})^{-1}(B))$ is connected. By a continuity argument [15, Lemma 5.1] this implies that $\Phi^{-1}(x)$ is connected. \square

We obtain Theorem 1.1 as a special case:

Proof of Theorem 1.1. Consider again the twisted moduli space for the r -holed sphere Σ_0^r , corresponding to $\kappa_i \in \text{Out}(G)$ with $\kappa_r \kappa_{r-1} \cdots \kappa_1 = 1$, as in Lemma 3.8. We had found that the moment map image consists of all (d_1, \dots, d_r) for which there exist elements $g_i \in G$ in the κ_i -twisted conjugacy class of d_i , such that $g_1 \cdots g_r = e$. Hence, by Theorem 4.4 the set (4) is a convex polytope. \square

REFERENCES

1. C. Agnihotri, S. and Woodward, *Eigenvalues of products of unitary matrices and quantum Schubert calculus*, Math. Res. Lett. **5** (1998), no. 6, 817–836.
2. A. Alekseev, H. Bursztyn, and E. Meinrenken, *Pure spinors on Lie groups*, Astérisque **327** (2009), 131–199.
3. A. Alekseev, A. Malkin, and E. Meinrenken, *Lie group valued moment maps*, J. Differential Geom. **48** (1998), no. 3, 445–495.
4. T. Baird, *Classifying spaces of twisted loop groups*, Preprint, 2013, arXiv:1312.7450.
5. P. Belkale, *Local systems on $\mathbb{P}^1 - S$ for S a finite set*, Compositio Math. **129** (2001), no. 1, 67–86.
6. P. Belkale and S. Kumar, *The multiplicative eigenvalue problem and deformed quantum cohomology*, Preprint, 2013, arXiv:1310.3191.
7. A. Berenstein and R. Sjamaar, *Coadjoint orbits, moment polytopes, and the Hilbert-Mumford criterion*, J. Amer. Math. Soc. **13** (2000), 433–466.
8. P. Boalch, *Quasi-Hamiltonian Geometry of Meromorphic Connections*, Duke Math. J. **139** (2007), 369–405.

9. ———, *Geometry and braiding of Stokes data; Fission and wild character varieties*, *Annals of Math.* **179** (2014), 301–365.
10. P. Boalch and D. Yamakawa, *Twisted wild character varieties*, Preprint, arXiv:1512.08091.
11. H. Bursztyn and M. Crainic, *Dirac structures, momentum maps, and quasi-Poisson manifolds*, *The breadth of symplectic and Poisson geometry*, *Progr. Math.*, vol. 232, Birkhäuser Boston, Boston, MA, 2005, pp. 1–40.
12. V. Guillemin and S. Sternberg, *A normal form for the moment map*, *Differential Geometric Methods in Mathematical Physics* (Jerusalem, 1982) (S. Sternberg, ed.), *Mathematical Physics Studies*, vol. 6, D. Reidel Publishing Company, Dordrecht, 1984, pp. 161–175.
13. V. Kac, *Infinite-dimensional Lie algebras*, second ed., Cambridge University Press, Cambridge, 1985.
14. A. Klyachko, *Stable bundles, representation theory and Hermitian operators*, *Selecta Math. (N.S.)* **4** (1998), no. 3, 419–445.
15. E. Lerman, E. Meinrenken, S. Tolman, and C. Woodward, *Non-abelian convexity by symplectic cuts*, *Topology* **37** (1998), 245–259.
16. C.-M. Marle, *Modèle d'action Hamiltonienne d'un groupe de Lie sur une variété symplectique*, *Rend. Sem. Mat. Univ. Politec. Torino* (1985), 227–251.
17. E. Meinrenken and C. Woodward, *Hamiltonian loop group actions and Verlinde factorization*, *J. Differential Geom.* **50** (1999), 417–470.
18. S. Mohr dieck, *Conjugacy classes of non-connected semisimple algebraic groups*, Universität Hamburg, 2000.
19. S. Mohr dieck and R. Wendt, *Integral conjugacy classes of compact Lie groups*, *Manuscripta Math.* **113** (2004), no. 4, 531–547.
20. N. Ressayre, *Geometric invariant theory and the generalized eigenvalue problem*, *Invent. Math.* **180** (2010), no. 2, 389–441.
21. T. A. Springer, *Twisted conjugacy in simply connected groups*, *Transform. Groups* **11** (2006), no. 3, 539–545.
22. C. Teleman and C. Woodward, *Parabolic bundles, products of conjugacy classes, and quantum cohomology*, *Ann. Inst. Fourier (Grenoble)* (2003), 713–748.
23. M. Vergne and M. Walter, *Inequalities for moment cones of finite-dimensional representations*, Preprint (2014). arXiv:1410.8144.

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